Galois characterization of Endoscopy for rational Siegel modular forms

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Abstract

We establish a relation between Galois reducibility and Endoscopy for genus 2 Siegel cusp forms which have rational eigenvalues and are unramified at 3.

1 The Theorem

Let f be a genus 2 Siegel cuspidal Hecke eigenform of weight $k > 2$ and $\mathbb{Q}_f$ the number field generated by its eigenvalues. It is well known that if f is not of Saito-Kurokawa type but it is "endoscopic" (i.e., it is in the image of the weak endoscopic lift) the compatible family of Galois representations $\{\rho_{f,\lambda}\}$ attached to f (constructed by Taylor, Laumon and Weissauer for any Siegel cusp form) will be reducible over $\mathbb{Q}_f$, with two 2-dimensional irreducible components.

In this note we will prove that the converse statement is true, for the case $\mathbb{Q}_f = \mathbb{Q}$. We will have to impose a local condition at 3 and will assume that the determinants are minimally ramified. More precisely, the result is:

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Theorem 1.1 *Let f be a genus 2 Siegel cusp form of weight $k > 2$ with $\mathbb{Q}_f = \mathbb{Q}$, such that the corresponding automorphic representation π_f has multiplicity one and $\pi_{f,3}$, its local component at 3, is unramified. Assume that the compatible family of Galois representations $\{\rho_{f,\ell}\}$ attached to f reduces (over $\mathbb{Q}$) as follows:*

$$\rho_{f,\ell} \simeq \sigma_{1,\ell} \oplus \sigma_{2,\ell} \qquad (1.1)$$

for every prime ℓ, where $\{\sigma_{1,\ell}\}$ and $\{\sigma_{2,\ell}\}$ are compatible families of 2-dimensional irreducible representations both with determinant χ^{2k-3}.
Then, f is endoscopic. More precisely, there exist two classical cuspidal modular forms f_1, f_2, of weights 2 and $2k-2$ (respectively) such that the family of representations $\{\sigma_{1,\ell} \otimes \chi^{2-k}\}$ is attached to f_1 and the family $\{\sigma_{2,\ell}\}$ is attached to f_2.

Remarks: 1- The irreducibility assumption of the 2-dimensional components is equivalent to assume that f is not of Saito-Kurokawa type.
2- Reducibility of the whole family $\{\rho_{f,\ell}\}$ as in the statement of the theorem is equivalent to a similar condition imposed only at a single prime ℓ, provided that $\ell > 4k-5$ and $\pi_{f,\ell}$ is unramified. This follows from a result of "existence of a family" proved in [D2].

2 Proof

Irreducibility of the 2-dimensional components implies that we are not in the Saito-Kurokawa case, and together with the multiplicity one assumption this implies (as proved by Weissauer) that the representations are pure, odd, and for every prime ℓ such that $\pi_{f,\ell}$ is unramified, the representations $\sigma_{1,\ell}$ and $\sigma_{2,\ell}$ are crystalline with Hodge-Tate weights $\{k-2, k-1\}$ and $\{0, 2k-3\}$, respectively.
Let us first show modularity of the family $\{\sigma_{1,\ell} \otimes \chi^{2-k}\}$. By assumption, $\pi_{f,3}$ is unramified, thus $\sigma_{1,3} \otimes \chi^{2-k}$ is a Barsotti-Tate representation, irreducible, odd, with rational coefficients, and unramified outside a finite set of primes. Then, applying a combination of modularity results of Diamond-Taylor-Wiles and Skinner-Wiles (as done in [D1] and [D2]) we conclude that $\sigma_{1,3} \otimes \chi^{2-k}$ is modular, and this gives modularity of the family $\{\sigma_{1,\ell} \otimes \chi^{2-k}\}$. The corresponding modular form f_1 must have weight 2 because for almost

every ℓ the representations in this family are Barsotti-Tate.
This argument "à la Wiles" can not be applied to $\sigma_{2,3}$ because, even if we again have Wiles' starting point (namely, we know that residually $\bar{\sigma}_{2,3}$ is either modular or reducible), the prime 3 is too small compared with the difference $2k-3$ of the Hodge-Tate weights to make the strategy workable.
To show modularity of the family $\{\sigma_{2,\ell}\}$ we will explode the fact that (1.1) is telling us that the representations $\sigma_{2,\ell}$ can be obtained by "substracting" a modular representation from another modular representation.
A key ingredient is a result recently proved by Weselmann (yet unpublished, but see [BWW] and [W]), which states that π_f has a weak lift to an automorphic representation π' of $\mathrm{GL}(4,\mathbb{A})$, where $\mathbb{A}$ are the rational adeles. Thus, by Cebotarev, the family $\{\rho_{f,\ell}\}$ is also attached to π'.
We want to apply a result of Jacquet and Shalika (which appears as theorem 3.3 in [T2]), in a similar way than what is done in [T2], section 533. We have from (1.1) the equality of L-functions:

$$L(\pi', s) = L(\sigma_{2,\ell}, s)L(f_1 \otimes \chi^{k-2}, s)$$

Observe that π' is the weak lift of π_f, but it is not necessarily cuspidal.
To conclude that $\sigma_{2,\ell}$ is modular, as in section 533 of [T2], we must find a prime ℓ such that $L(\sigma_{2,\ell}^* \otimes \sigma_{2,\ell}, s)$ has a simple pole at $s = 1$, because in that case the result of Jacquet and Shalika implies $\sigma_{2,\ell} \simeq \sigma_{\pi_i,\ell}$, where π_i is one of the cuspidal constituents of π'. Then, it only remains to find a prime satisfying this condition.
Take $\ell > 4k-5$ such that the local components of π' and π_f at ℓ are unramified. For such a prime ℓ the representation $\sigma_{2,\ell}$ is crystalline with Hodge-Tate weights $\{0, 2k-3\}$ and the main result of [T1] implies that there exists a totally real number field F such that the restriction of $\sigma_{2,\ell}$ to the Galois group of F is modular, i.e., it agrees with the Galois representation attached to a Hilbert modular form over F.
But, as explained in [T2], section 533, precisely from this potentially modular property (and the fact that it is preserved after solvable base change) one can deduce that $L(\sigma_{2,\ell}^* \otimes \sigma_{2,\ell}, s)$ does have a simple pole at $s = 1$, as we wanted. This shows modularity of the family $\{\sigma_{2,\ell}\}$ and it is clear from its Hodge-Tate decomposition that it corresponds to a modular form of weight $2k-2$. We conclude that the Siegel cusp form f is endoscopic.

3 Bibliography

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